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Yang–Baxter equation for the R-matrix of the 1D $SU(n)$ Hubbard model

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Abstract
Based on the tetrahedral Zamolodchikov algebra, we prove the Yang–Baxter equation for the R-matrix of the 1D $SU(n)$ Hubbard model. Furthermore, we present generalizations of the model.

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1. Introduction

The Hubbard model is one of the significant models in the study of strongly correlated electronic systems which might play an enlightening role in understanding the mysteries of high-T_C superconductivity. The 1D Hubbard model also favours a lot of properties of integrable models in non-perturbative quantum field theory and mathematical physics. Since Lieb and Wu [1] solved the 1D Hubbard model by the Bethe ansatz method in 1968, based on their results (Lieb and Wu's Bethe ansatz equations), many works [2–18] have been extensively carried out to clarify the physical properties of this model. Although there was considerable research on the Hubbard model, the integrability was completed in 1986 by Shastry [19, 20] in both boson and fermion graded versions. However, the Yang–Baxter equation for the R-matrix given by Shastry was proved in 1995 by Shiroishi and Wadati [21–23] and a generalization of Shastry's bilayer vertex model was also presented in [21]. Moreover, the eigenvalue of the transfer matrix related to the Hubbard model was suggested in [19] and proved through different methods in [24, 25] (for a review see [26]).

Based on the knowledge of Lie algebra, Maassarani and Mathieu succeeded in constructing the Hamiltonian of the $SU(n)$ XX model and showed its integrability [27]. Considering two coupled $SU(n)$ XX models, by using Shastry's method, Maassarani constructed the $SU(n)$ Hubbard model [28] and found the related R-matrix which ensures the integrability of the 1D $SU(n)$ Hubbard model [29]. (It was also proved for $n = 3, 4$ by Martins [30], and for general n in terms of a Lax pair formalism by Yue and Sasaki [31].) This generalization is different from the other integrable $SU(n)$ generalization of the Hubbard model (suggested by Choy [32] and Haldane [33], see also [34–36]). By using the Bethe ansatz method, the exact solution of the $SU(3)$ Hubbard model was given in [37]. But the Yang–Baxter equation for the given R-matrix was not proved.

The main purpose of the present paper is to prove the Yang–Baxter equation for the R-matrix of the 1D $SU(n)$ Hubbard model following the method suggested in [21]. In section 2 we review the model and its integrability. We present the L-operator and the R-matrix of the model and formulate the Yang–Baxter relation. In section 3 we construct the tetrahedral Zamolodchikov algebra related to the $SU(n)$ Hubbard model. The Yang–Baxter equation for the corresponding R-matrix was proved in section 4 and we also present a generalization of the model in this section. In section 5 we make some concluding remarks.

2. The 1D $SU(n)$ Hubbard model and its integrability

The Hamiltonian of the 1D $SU(n)$ Hubbard model is

$$H = \sum_{k=1}^{L}\sum_{\alpha=1}^{n-1}\left(E^{n\alpha}_{\sigma,k}E^{\alpha n}_{\sigma,k+1} + E^{\alpha n}_{\sigma,k}E^{n\alpha}_{\sigma,k+1} + E^{n\alpha}_{\tau,k}E^{\alpha n}_{\tau,k+1} + E^{\alpha n}_{\tau,k}E^{n\alpha}_{\tau,k+1}\right) + \frac{Un^2}{4}\sum_{k=1}^{L} C^{(\sigma)}_k C^{(\tau)}_k \tag{1}$$

where U is the Coulomb coupling constant, and $E^{\alpha\beta}_{a,k}$ $(a = \sigma, \tau)$ is a matrix with a one at row α and column β and zeros otherwise:

$$\left(E^{\alpha\beta}_{a,k}\right)_{lm} = \delta^\alpha_l \delta^\beta_m.$$

The subscripts a and k stand for two different E operators at the kth site $(k = 1, \ldots, L)$. The $n \times n$ diagonal matrix $C^{(a)}_k$ is defined by $C^{(a)}_k = \sum_{\alpha<n} E^{\alpha\alpha}_{a,k} - E^{nn}_{a,k}$. We also assume the periodic boundary condition, $E^{\alpha\beta}_{a,k+L} = E^{\alpha\beta}_{a,k}$.

In this model, the system has two types of particles named σ and τ respectively, and each particle can occupy $(n-1)$ possible states. The same type of particles cannot appear in one site, but two different types of particles can occupy the same site. We denote $|n\rangle_j$ as the vacuum state of the jth site, $|1\rangle_j, |2\rangle_j, \ldots, |n-1\rangle_j$ as the $(n-1)$ possible one particle states of the jth site. On the following basis,

$$|1\rangle_j = \begin{pmatrix}1\\0\\\vdots\\0\end{pmatrix}_j, |2\rangle_j = \begin{pmatrix}0\\1\\\vdots\\0\end{pmatrix}_j, \ldots, |n-1\rangle_j = \begin{pmatrix}0\\\vdots\\1\\0\end{pmatrix}_j, |n\rangle_j = \begin{pmatrix}0\\\vdots\\0\\1\end{pmatrix}_j$$

it could be easily proved that $E^{\alpha n}_j|n\rangle_j = |\alpha\rangle_j$, $E^{n\alpha}_j|n\rangle_j = 0$, $E^{n\alpha}_j|\alpha\rangle_j = |n\rangle_j$, $E^{\alpha n}_j|\alpha\rangle_j = 0$. This means that the operators $E^{\alpha n}$ and $E^{n\alpha}$ can be interpreted as the particle creation and annihilation operators, respectively. $E^{\alpha n}_j$ creates a $|\alpha\rangle_j$ state particle over the vacuum state $|n\rangle_j$ of the jth site, and $E^{n\alpha}_j$ annihilates a $|\alpha\rangle_j$ state particle to the vacuum state of the jth site. Under this representation of the particle states, we realized that there cannot be more than two different particles on one site.

The $SU(n)$ Hubbard model is constructed by considering two coupled $SU(n)$ XX models, so the Hamiltonian (1) consists of two $SU(n)$ XX models with an interaction term between them. The Hamiltonian of the $SU(n)$ XX model is

$$H_{\mathrm{XX}} = \sum_{k=1}^{L}\sum_{\alpha=1}^{n-1}\left(E^{n\alpha}_k E^{\alpha n}_{k+1} + E^{\alpha n}_k E^{n\alpha}_{k+1}\right) \tag{2}$$

and the corresponding R-matrix is

$$R(\lambda) = a(\lambda)\left[E^{nn}\otimes E^{nn} + \sum_{\alpha,\beta<n} E^{\alpha\beta}\otimes E^{\beta\alpha}\right] + b(\lambda)\sum_{\alpha<n}(xE^{nn}\otimes E^{\alpha\alpha} + x^{-1}E^{\alpha\alpha}\otimes E^{nn}) + c(\lambda)\sum_{\alpha<n}(E^{n\alpha}\otimes E^{\alpha n} + E^{\alpha n}\otimes E^{n\alpha}) \quad (3)$$

where $x = e^{i\delta}$ and $a(\lambda) = \cos(\lambda), b(\lambda) = \sin(\lambda), c(\lambda) = 1$. The functions $a(\lambda), b(\lambda), c(\lambda)$ satisfy the free-fermion relation, $a^2(\lambda) + b^2(\lambda) = c^2(\lambda)$.

The R-matrix of the $SU(n)$ XX model satisfies regularity property $R(0) = P$, unitarity condition $R_{12}(\lambda)R_{21}(-\lambda) = \cos^2(\lambda)Id$ and the Yang–Baxter equation (YBE)

$$R_{31}(\lambda_1)R_{32}(\lambda_2)R_{12}(\lambda_2-\lambda_1) = R_{12}(\lambda_2-\lambda_1)R_{32}(\lambda_2)R_{31}(\lambda_1) \quad (4)$$

where P is a permutation operator on the tensor product of two n-dimensional spaces. It is easy to verify that this also satisfies a decorated Yang–Baxter equation (DYBE)

$$R_{31}(\lambda_1)R_{32}(\lambda_2)C_2R_{12}(\lambda_2+\lambda_1) = R_{12}(\lambda_2+\lambda_1)C_2R_{32}(\lambda_2)R_{31}(\lambda_1). \quad (5)$$

Here C_2 is the C-matrix for the second space which is defined as $C_2 = 1\otimes\left(\sum_{\alpha=1}^{n-1}E^{\alpha\alpha} - E^{nn}\right)$.

Considering the two $SU(n)$ XX models without interaction, the R-matrix is given by

$$\bar{R}_{ij}(\lambda) = R_{ij}^{(\sigma)}(\lambda)R_{ij}^{(\tau)}(\lambda). \quad (6)$$

Here $R_{ij}^{(\sigma)}(\lambda)$ and $R_{ij}^{(\tau)}(\lambda)$ denote the R-matrices of two $SU(n)$ XX models. Since both $R_{ij}^{(\sigma)}(\lambda)$ and $R_{ij}^{(\tau)}(\lambda)$ satisfy the YBE and DYBE, the product $\bar{R}_{ij}(\lambda)$ also satisfies the YBE,

$$\bar{R}_{31}(\lambda_1)\bar{R}_{32}(\lambda_2)\bar{R}_{12}(\lambda_2-\lambda_1) = \bar{R}_{12}(\lambda_2-\lambda_1)\bar{R}_{32}(\lambda_2)\bar{R}_{31}(\lambda_1) \quad (7)$$

and DYBE,

$$\bar{R}_{31}(\lambda_1)\bar{R}_{32}(\lambda_2)C_2^{(\sigma)}C_2^{(\tau)}\bar{R}_{12}(\lambda_2+\lambda_1) = \bar{R}_{12}(\lambda_2+\lambda_1)C_2^{(\sigma)}C_2^{(\tau)}\bar{R}_{32}(\lambda_2)\bar{R}_{31}(\lambda_1). \quad (8)$$

A linear combination of (7) and (8) yields

$$\bar{R}_{31}(\lambda_1)\bar{R}_{32}(\lambda_2)\left\{\alpha\bar{R}_{12}(\lambda_2-\lambda_1) + \beta C_2^{(\sigma)}C_2^{(\tau)}\bar{R}_{12}(\lambda_2+\lambda_1)\right\} = \left\{\alpha\bar{R}_{12}(\lambda_2-\lambda_1) + \beta\bar{R}_{12}(\lambda_2+\lambda_1)C_2^{(\sigma)}C_2^{(\tau)}\right\}\bar{R}_{32}(\lambda_2)\bar{R}_{31}(\lambda_1). \quad (9)$$

Here α and β are combination coefficients and arbitrary.

For the $SU(n)$ Hubbard model, the two coupled $SU(n)$ XX models, we look for a solution of the Yang–Baxter relation (YBR),

$$\mathcal{L}_{31}(\lambda_1)\mathcal{L}_{32}(\lambda_2)R_{12}^h(\lambda_1,\lambda_2) = R_{12}^h(\lambda_1,\lambda_2)\mathcal{L}_{32}(\lambda_2)\mathcal{L}_{31}(\lambda_1) \quad (10)$$

in the form

$$R_{12}^h(\lambda_1,\lambda_2) = \alpha\bar{R}_{12}(\lambda_2-\lambda_1) + \beta\bar{R}_{12}(\lambda_2+\lambda_1)C_2^{(\sigma)}C_2^{(\tau)} \quad (11)$$

$$\mathcal{L}_{ij}(\lambda) = \bar{R}_{ij}(\lambda)\exp\left\{h(\lambda)C_j^{(\sigma)}C_j^{(\tau)}\right\}. \quad (12)$$

Comparing equation (9) with the YBR (10), we get a relation

$$I_1(\lambda_1)I_2(\lambda_2)R_{12}^h(\lambda_1,\lambda_2)I_1^{-1}(\lambda_1)I_2^{-1}(\lambda_2) = \alpha\bar{R}_{12}(\lambda_2-\lambda_1) + \beta C_2^{(\sigma)}C_2^{(\tau)}\bar{R}_{12}(\lambda_2+\lambda_1) \quad (13)$$

where

$$I_j(\lambda) = \exp\left\{h(\lambda)C_j^{(\sigma)}C_j^{(\tau)}\right\}. \quad (14)$$

From (13), we have

$$\begin{aligned}\frac{\beta a(\lambda_2+\lambda_1)c(\lambda_2+\lambda_1)}{\alpha a(\lambda_2-\lambda_1)c(\lambda_2-\lambda_1)} &= \tanh(h(\lambda_2)-h(\lambda_1))\\ \frac{\beta b(\lambda_2+\lambda_1)c(\lambda_2+\lambda_1)}{\alpha b(\lambda_2-\lambda_1)c(\lambda_2-\lambda_1)} &= \tanh(h(\lambda_2)+h(\lambda_1))\end{aligned} \tag{15}$$

which give the ratio of α and β and constraints on $h(\lambda_1)$ and $h(\lambda_2)$. The constraints can be written in more explicit form [29],

$$\frac{\sinh(h(\lambda_1))}{\sin(2\lambda_1)} = \frac{\sinh(h(\lambda_2))}{\sin(2\lambda_2)} = \frac{n^2U}{4}. \tag{16}$$

Now we have obtained the R-matrix of the 1D $SU(n)$ Hubbard model [29],

$$\begin{aligned}R^h_{12}(\lambda_1,\lambda_2) = {}& R^{(\sigma)}_{12}(\lambda_2-\lambda_1)R^{(\tau)}_{12}(\lambda_2-\lambda_1) + \frac{\cos(\lambda_2-\lambda_1)}{\cos(\lambda_2+\lambda_1)}\tanh(h(\lambda_2)-h(\lambda_1))\\ &\times R^{(\sigma)}_{12}(\lambda_2+\lambda_1)R^{(\tau)}_{12}(\lambda_2+\lambda_1)C^{(\sigma)}_2C^{(\tau)}_2\end{aligned} \tag{17}$$

which satisfies the YBR (10). This R-matrix depends not only on the difference of the spectral parameters $\lambda_2-\lambda_1$, but also on the sum of the spectral parameters $\lambda_2+\lambda_1$. This non-additive property allows us to generalize the Hamiltonian of the 1D $SU(n)$ Hubbard model (see section 4).

The monodromy matrix of the model can be defined as

$$\mathcal{T}_a(\lambda) = \mathcal{L}_{La}(\lambda)\mathcal{L}_{L-1a}(\lambda)\cdots\mathcal{L}_{1a}(\lambda). \tag{18}$$

From the YBR (10) we know that the monodromy matrix satisfies the global Yang–Baxter relation:

$$\mathcal{T}_1(\lambda_1)\mathcal{T}_2(\lambda_2)R^h_{12}(\lambda_1,\lambda_2) = R^h_{12}(\lambda_1,\lambda_2)\mathcal{T}_2(\lambda_2)\mathcal{T}_1(\lambda_1). \tag{19}$$

The corresponding transfer matrix is defined by

$$\tau(\lambda) = \mathrm{Tr}_a[\mathcal{T}_a(\lambda)] \tag{20}$$

and from (19) can easily be proved the existence of a commuting family of transfer matrices

$$[\tau(\lambda_1),\tau(\lambda_2)] = 0. \tag{21}$$

Then the integrability of the model is proved.

Using the relations $h(0)=0$ and $h'(0)=\frac{n^2U}{4}$ we can obtain the Hamiltonian of the 1D $SU(n)$ Hubbard model (1):

$$\begin{aligned}H &= \frac{\mathrm{d}}{\mathrm{d}\lambda}\ln\tau(\lambda)|_{\lambda=0} = \tau^{-1}(0)\frac{\mathrm{d}}{\mathrm{d}\lambda}\tau(\lambda)|_{\lambda=0}\\ &= \sum_{k=1}^{L}\sum_{\alpha=1}^{n-1}\left(E^{n\alpha}_{\sigma,k}E^{\alpha n}_{\sigma,k+1} + E^{\alpha n}_{\sigma,k}E^{n\alpha}_{\sigma,k+1} + E^{n\alpha}_{\tau,k}E^{\alpha n}_{\tau,k+1} + E^{\alpha n}_{\tau,k}E^{n\alpha}_{\tau,k+1}\right)\\ &\quad + \frac{Un^2}{4}\sum_{k=1}^{L}C^{(\sigma)}_kC^{(\tau)}_k.\end{aligned} \tag{22}$$

3. Tetrahedral Zamolodchikov algebra

In the above section, we have shown the integrability of the 1D $SU(n)$ Hubbard model. It is natural to expect that the R-matrix (17) itself satisfies the Yang–Baxter equation (YBE):

$$R^h_{31}(\lambda_3,\lambda_1)R^h_{32}(\lambda_3,\lambda_1)R^h_{12}(\lambda_1,\lambda_1) = R^h_{12}(\lambda_1,\lambda_2)R^h_{32}(\lambda_3,\lambda_2)R^h_{31}(\lambda_3,\lambda_1). \tag{23}$$

In the $SU(2)$ case, the YBE of the R-matrix was proved in [21] by using the tetrahedral Zamolodchikov algebra (TZA) [23, 38]. In this section, we construct the TZA related to the $SU(n)$ Hubbard model following [21]. It is the extension of Korepanov's result [39, 40] to the $SU(n)$ case.

The TZA is defined by the following set of relations,

$$\mathcal{L}^a_{12}\mathcal{L}^b_{32}\mathcal{L}^c_{31} = \sum_{def} S^{abc}_{def}\mathcal{L}^f_{31}\mathcal{L}^e_{32}\mathcal{L}^d_{12}. \tag{24}$$

where $a, b, \ldots, f = 0, 1$ and S^{abc}_{def} are some scalar coefficients.

We take $\mathcal{L}^0_{jk}$ and $\mathcal{L}^1_{jk}$ as follows,

$$\mathcal{L}^0_{jk} = R_{jk}(\lambda_k - \lambda_j) \qquad \mathcal{L}^1_{jk} = R_{jk}(\lambda_k + \lambda_j)C_k \tag{25}$$

where $R_{jk}(\lambda)$ is the R-matrix of the $SU(n)$ XX model as before. Then we could find the following relations which give the TZA (24),

$$\mathcal{L}^0_{12}\mathcal{L}^0_{32}\mathcal{L}^0_{31} = \mathcal{L}^0_{31}\mathcal{L}^0_{32}\mathcal{L}^0_{12} \qquad \mathcal{L}^0_{12}\mathcal{L}^1_{32}\mathcal{L}^1_{31} = \mathcal{L}^1_{31}\mathcal{L}^1_{32}\mathcal{L}^0_{12} \tag{26}$$

$$\mathcal{L}^1_{12}\mathcal{L}^1_{32}\mathcal{L}^0_{31} = \mathcal{L}^0_{31}\mathcal{L}^1_{32}\mathcal{L}^1_{12} \qquad \mathcal{L}^1_{12}\mathcal{L}^0_{32}\mathcal{L}^1_{31} = \mathcal{L}^1_{31}\mathcal{L}^0_{32}\mathcal{L}^1_{12} \tag{27}$$

$$\mathcal{L}^1_{12}\mathcal{L}^1_{32}\mathcal{L}^1_{31} = S^{111}_{001}\mathcal{L}^1_{31}\mathcal{L}^0_{32}\mathcal{L}^0_{12} + S^{111}_{010}\mathcal{L}^1_{31}\mathcal{L}^0_{32}\mathcal{L}^1_{12} + S^{111}_{100}\mathcal{L}^0_{31}\mathcal{L}^0_{32}\mathcal{L}^1_{12} \tag{28}$$

$$\mathcal{L}^0_{12}\mathcal{L}^0_{32}\mathcal{L}^1_{31} = S^{001}_{111}\mathcal{L}^1_{31}\mathcal{L}^1_{32}\mathcal{L}^1_{12} + S^{001}_{100}\mathcal{L}^0_{31}\mathcal{L}^0_{32}\mathcal{L}^1_{12} + S^{001}_{010}\mathcal{L}^0_{31}\mathcal{L}^1_{32}\mathcal{L}^0_{12} \tag{29}$$

$$\mathcal{L}^0_{12}\mathcal{L}^1_{32}\mathcal{L}^0_{31} = S^{010}_{111}\mathcal{L}^1_{31}\mathcal{L}^1_{32}\mathcal{L}^1_{12} + S^{010}_{100}\mathcal{L}^0_{31}\mathcal{L}^0_{32}\mathcal{L}^1_{12} + S^{010}_{001}\mathcal{L}^1_{31}\mathcal{L}^0_{32}\mathcal{L}^0_{12} \tag{30}$$

$$\mathcal{L}^1_{12}\mathcal{L}^0_{32}\mathcal{L}^0_{31} = S^{100}_{111}\mathcal{L}^1_{31}\mathcal{L}^1_{32}\mathcal{L}^1_{12} + S^{100}_{010}\mathcal{L}^0_{31}\mathcal{L}^1_{32}\mathcal{L}^0_{12} + S^{100}_{001}\mathcal{L}^1_{31}\mathcal{L}^0_{32}\mathcal{L}^0_{12} \tag{31}$$

where the coefficients S^{abc}_{def} are given by

$$\begin{aligned}
S^{111}_{001} &= \frac{\sin(\lambda_2+\lambda_1)\cos(\lambda_2+\lambda_3)}{\cos(\lambda_2-\lambda_1)\sin(\lambda_2-\lambda_3)} & S^{111}_{010} &= -\frac{\sin(\lambda_2+\lambda_1)\sin(\lambda_1+\lambda_3)}{\cos(\lambda_2-\lambda_1)\cos(\lambda_1-\lambda_3)} \\
S^{111}_{100} &= -\frac{\sin(\lambda_1+\lambda_3)\cos(\lambda_2+\lambda_3)}{\cos(\lambda_1-\lambda_3)\sin(\lambda_2-\lambda_3)} & S^{001}_{111} &= \frac{\sin(\lambda_2-\lambda_1)\cos(\lambda_2-\lambda_3)}{\cos(\lambda_2+\lambda_1)\sin(\lambda_2+\lambda_3)} \\
S^{001}_{100} &= \frac{\sin(\lambda_2-\lambda_1)\sin(\lambda_1+\lambda_3)}{\cos(\lambda_2+\lambda_1)\cos(\lambda_1-\lambda_3)} & S^{001}_{010} &= \frac{\sin(\lambda_1+\lambda_3)\cos(\lambda_2-\lambda_3)}{\cos(\lambda_1-\lambda_3)\cos(\lambda_2+\lambda_3)} \\
S^{010}_{111} &= \frac{\sin(\lambda_2-\lambda_1)\sin(\lambda_1-\lambda_3)}{\cos(\lambda_2+\lambda_1)\cos(\lambda_1+\lambda_3)} & S^{010}_{100} &= \frac{\sin(\lambda_2-\lambda_1)\cos(\lambda_2+\lambda_3)}{\cos(\lambda_2+\lambda_1)\sin(\lambda_2-\lambda_3)} \\
S^{010}_{001} &= \frac{\sin(\lambda_1-\lambda_3)\cos(\lambda_2+\lambda_3)}{\cos(\lambda_1+\lambda_3)\sin(\lambda_2-\lambda_3)} & S^{100}_{111} &= -\frac{\sin(\lambda_1-\lambda_3)\cos(\lambda_2-\lambda_3)}{\cos(\lambda_1+\lambda_3)\sin(\lambda_2+\lambda_3)} \\
S^{100}_{010} &= \frac{\sin(\lambda_2+\lambda_1)\cos(\lambda_2-\lambda_3)}{\cos(\lambda_2-\lambda_1)\sin(\lambda_2+\lambda_3)} & S^{100}_{001} &= -\frac{\sin(\lambda_2+\lambda_1)\sin(\lambda_1-\lambda_3)}{\cos(\lambda_2-\lambda_1)\cos(\lambda_1+\lambda_3)}.
\end{aligned} \tag{32}$$

Equations (26) and (27) are equivalent to the YBE (7) and DYBE (8) respectively. In this sense, the TZA (24) can be regarded as a generalization of the YBE and DYBE.

It is important to note that the products $\mathcal{L}^a_{12}\mathcal{L}^b_{32}\mathcal{L}^c_{31}$ are not linearly independent as operators acting on $V_1 \otimes V_2 \otimes V_3$ and they satisfy the following relations,

$$\mathcal{L}^0_{12}\mathcal{L}^0_{32}\mathcal{L}^0_{31} = x_0\mathcal{L}^0_{12}\mathcal{L}^1_{32}\mathcal{L}^1_{31} + y_0\mathcal{L}^1_{12}\mathcal{L}^0_{32}\mathcal{L}^1_{31} + z_0\mathcal{L}^1_{12}\mathcal{L}^1_{32}\mathcal{L}^0_{31} \tag{33}$$

$$\mathcal{L}^1_{12}\mathcal{L}^1_{32}\mathcal{L}^1_{31} = x_1\mathcal{L}^1_{12}\mathcal{L}^0_{32}\mathcal{L}^0_{31} + y_1\mathcal{L}^0_{12}\mathcal{L}^1_{32}\mathcal{L}^0_{31} + z_1\mathcal{L}^0_{12}\mathcal{L}^0_{32}\mathcal{L}^1_{31} \tag{34}$$

with

$$x_0 = -\frac{\cos(\lambda_1-\lambda_3)\sin(\lambda_2-\lambda_3)}{\cos(\lambda_1+\lambda_3)\sin(\lambda_2+\lambda_3)} \qquad y_0 = \frac{\cos(\lambda_2-\lambda_1)\cos(\lambda_1-\lambda_3)}{\cos(\lambda_2+\lambda_1)\cos(\lambda_1+\lambda_3)} \tag{35}$$

$$z_0 = \frac{\cos(\lambda_2-\lambda_1)\sin(\lambda_2-\lambda_3)}{\cos(\lambda_2+\lambda_1)\sin(\lambda_2+\lambda_3)} \qquad x_1 = -\frac{\cos(\lambda_1+\lambda_3)\sin(\lambda_2+\lambda_3)}{\cos(\lambda_1-\lambda_3)\sin(\lambda_2-\lambda_3)} \tag{36}$$

$$y_1 = \frac{\cos(\lambda_2+\lambda_1)\cos(\lambda_1+\lambda_3)}{\cos(\lambda_2-\lambda_1)\cos(\lambda_1-\lambda_3)} \qquad z_1 = \frac{\sin(\lambda_2+\lambda_3)\cos(\lambda_2+\lambda_1)}{\sin(\lambda_2-\lambda_3)\cos(\lambda_2-\lambda_1)}. \tag{37}$$

From these relations, we know that the linear space spanned by the products $\mathcal{L}^a_{12}\mathcal{L}^b_{32}\mathcal{L}^c_{31}$ is six dimensional.

4. The Yang–Baxter equation for the *R*-matrix of the $SU(n)$ Hubbard model

In this section, we prove the Yang–Baxter equation for the R-matrix of the 1D $SU(n)$ Hubbard model (23).

Taking into account the form of the R-matrix (17), we look for a solution of the YBE (23) in the following form,

$$\begin{aligned} R^h_{jk}(\lambda_j,\lambda_k) &= R^{(\sigma)}_{jk}(\lambda_k-\lambda_j)R^{(\tau)}_{jk}(\lambda_k-\lambda_j) + \alpha_{jk}R(\sigma)_{jk}(\lambda_k+\lambda_j)C^{(\sigma)}_k R^{(\tau)}_{jk}(\lambda_k+\lambda_j)C^{(\tau)}_k \\ &= \mathcal{L}^{0(\sigma)}_{jk}\mathcal{L}^{0(\tau)}_{jk} + \alpha_{jk}\mathcal{L}^{1(\sigma)}_{jk}\mathcal{L}^{1(\tau)}_{jk} \end{aligned} \tag{38}$$

where $\mathcal{L}^0_{jk}$ and $\mathcal{L}^1_{jk}$ have been defined in (25). If $\alpha_{jk}=0$, the R-matrix satisfies the YBE (23) in a trivial way. Now we look for a non-trivial solution. Substituting expression (38) into the Yang–Baxter equation (23), by means of the tetrahedral Zamolodchikov algebra and relations (33) and (34), we could find that α_{jk} must satisfy the following condition:

$$\begin{aligned} \alpha_{12}\sin 2(\lambda_1+\lambda_2) + \alpha_{31}\sin 2(\lambda_1+\lambda_3) &= \alpha_{32}\sin 2(\lambda_2+\lambda_3) \\ &= \frac{1}{\alpha_{31}}\sin 2(\lambda_3-\lambda_1) + \frac{1}{\alpha_{12}}\sin 2(\lambda_2-\lambda_1). \end{aligned} \tag{39}$$

If we take

$$\alpha_{jk} = \frac{\cos(\lambda_k-\lambda_j)}{\cos(\lambda_k+\lambda_j)}\tanh(h(\lambda_k)-h(\lambda_j)) \tag{40}$$

and impose the constraints

$$\frac{\sinh(h(\lambda_j))}{\sin(2\lambda_j)} = \frac{n^2U}{4} \qquad (j=1,2,3) \tag{41}$$

then condition (39) is satisfied. This proves the Yang–Baxter equation for the R-matrix of the 1D $SU(n)$ Hubbard model.

Besides the YBE (23), the R-matrix (17) also has the following properties,

$$R^h_{jk}(0,\lambda) = \frac{1}{\cosh(h(\lambda))}\mathcal{L}_{jk}(\lambda) \tag{42}$$

$$R^h_{jk}(\lambda_0,\lambda_0) = \mathcal{P}_{jk} \tag{43}$$

$$R^h_{jk}(\lambda_j,\lambda_k)R^h_{kj}(\lambda_k,\lambda_j) = \rho(\lambda_j,\lambda_k)Id \tag{44}$$

where

$$\rho(\lambda_j, \lambda_k) = \cos^2(\lambda_k - \lambda_j)\{\cos^2(\lambda_k - \lambda_j) - \tanh^2(h(\lambda_k) - h(\lambda_j))\} \tag{45}$$

and the permutation operator is defined as

$$\mathcal{P}_{jk} = \mathcal{P}_{jk}^{(\sigma)}\mathcal{P}_{jk}^{(\tau)}. \tag{46}$$

The Yang–Baxter equation (23) implies a more general inhomogeneous model as

$$\mathcal{T}_a(\lambda, \{\lambda_j\}) = R_{La}^h(\lambda, \lambda_N) R_{L-1a}^h(\lambda, \lambda_{L-1}) \cdots R_{1a}^h(\lambda, \lambda_1) \tag{47}$$

where λ_j $(j = 1, 2, \ldots, L)$ are the inhomogeneous parameters obeying the constraints

$$\frac{\sinh(2h(\lambda_j))}{\sin(2\lambda_j)} = \frac{n^2 U}{4} \qquad (j = 1, 2, \ldots, L). \tag{48}$$

From the Yang–Baxter equation (23), we can obtain the global Yang–Baxter relation,

$$\mathcal{T}_1(\lambda, \{\lambda_j\})\mathcal{T}_2(\mu, \{\lambda_j\}) R_{12}^h(\lambda, \mu) = R_{12}^h(\lambda, \mu)\mathcal{T}_2(\mu, \{\lambda_j\})\mathcal{T}_1(\lambda, \{\lambda_j\}) \tag{49}$$

which leads to the commutativity

$$[\tau(\lambda, \{\lambda_j\}), \tau(\mu, \{\lambda_j\})] = 0 \tag{50}$$

where $\tau(\lambda, \{\lambda_j\})$ is the transfer matrix of the model

$$\tau(\lambda, \{\lambda_j\}) = \mathrm{Tr}_a \mathcal{T}_a(\lambda, \{\lambda_j\}). \tag{51}$$

The corresponding Hamiltonian is defined as the logarithmic derivative of the transfer matrix under all inhomogeneous parameters $\lambda_j = \lambda_0$ $(j = 1, 2, \ldots, L)$,

$$\begin{aligned} H_{\lambda_0} &= \frac{\mathrm{d}}{\mathrm{d}\lambda} \ln \tau(\lambda, \{\lambda_j = \lambda_0\})|_{\lambda=\lambda_0} = \tau^{-1}(\lambda_0, \{\lambda_j = \lambda_0\}) \frac{\mathrm{d}}{\mathrm{d}\lambda} \tau(\lambda, \{\lambda_j = \lambda_0\}) \\ &= \sum_{j=1}^{L} \sum_{\alpha<n} \left(E_{\sigma j}^{n\alpha} E_{\sigma j+1}^{\alpha n} + E_{\sigma j}^{\alpha n} E_{\sigma j+1}^{n\alpha} + E_{\tau j}^{n\alpha} E_{\tau j+1}^{\alpha n} + E_{\tau j}^{\alpha n} E_{\tau j+1}^{n\alpha} \right) \\ &\quad + \frac{n^2 U}{4\cosh(2h(\lambda_0))} \sum_{j=1}^{L} B_{jj+1}^{(\sigma)} B_{jj+1}^{(\tau)} \end{aligned} \tag{52}$$

where

$$\begin{aligned} B_{jj+1} &= \cos(2\lambda_0) \left(-E_j^{nn} E_{j+1}^{nn} + \sum_{\alpha,\beta<n} E_j^{\alpha\alpha} E_{j+1}^{\beta\beta} \right) + \sin(2\lambda_0) \sum_{\alpha<n} \left(E_j^{n\alpha} E_{j+1}^{\alpha n} + E_j^{\alpha n} E_{j+1}^{n\alpha} \right) \\ &\quad + \sum_{\alpha<n} \left(-E_j^{nn} E_{j+1}^{\alpha\alpha} + E_j^{\alpha\alpha} E_{j+1}^{nn} \right). \end{aligned} \tag{53}$$

The arbitrariness of the parameter λ_0 comes from the non-additive property of the spectral parameters. If we take $\lambda_0 = 0$, this new Hamiltonian reduces to (1).

Thus, we have obtained a new 1D $SU(n)$ Hubbard by the Yang–Baxter equation of the R-matrix (23).

5. Conclusions

In this paper, we have proved the R-matrix of the 1D $SU(n)$ Hubbard model satisfying the Yang–Baxter equation. We note that the tetrahedral Zamolodchikov algebra plays an essential role in the proof.

In most lattice systems, the existence of the R-matrix ensures the integrability and the R-matrix is isomorphic to the L-operator. Thus, the Yang–Baxter equation is a consequence of the Yang–Baxter relation $R_{12}L_1L_2 = L_2L_1R_{12}$. But, for the Hubbard model, the situation is quite different. The R-matrix cannot be obtained from the L-operator, even if we limit to $SU(2)$ Hubbard model [19, 20]. The R-matrix satisfying the Yang–Baxter equation together with the L-operator constitutes the complete proof of the integrability.

For the $SU(n)$ Hubbard model, the R-matrix is not isomorphic to the L-operator. This provides a method to construct a new kind of integrable system by considering the R-matrix as an L-operator (fundamental representation of the same algebra). The general representation can be obtained by fusing fundamental representations (R-matrices). Therefore, one can get the full representation of the algebra in principle.

In the present paper, we have derived a new Hamiltonian (52) from the R-matrix. The last term introduces a new kind of interaction. This Hamiltonian is quite different from the original one (1). It also gives rise to the question of how to find the eigenvalues of this Hamiltonian (52). This question will be considered in a later publication.

In the derivation of equation (52), we have assumed that all the parameters λ_j are equal to λ_0. This is not necessary. A different choice will give a different Hamiltonian. On the other hand, one can consider the R-matrix as a Boltzmann weight in statistical mechanics. This provides a inhomogeneous lattice statistical model. The partition function could be derived in a straightforward way.

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